Solution of Fluid Mechanics sample exam 2

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May 2023

Question 1

The correct solution is **c**. Consider the momentum equation:

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_{CV} \vec{v}\rho d\forall + \int_{CS} \vec{v}\rho(\vec{V} \cdot d\vec{A}). \tag{1}$$

The time derivative term on the RHS is zero in the steady flow. With the uniform flow assumption, the integration of momentum flux over the control surface (the second term on the RHS) can be simplified to:

$$M_x = \rho V_x \int_{CS} \vec{V} \cdot d\vec{A} = \rho V_x Q. \tag{2}$$

Question 2

We build the control volume as in Fig 1. Consider the momentum equation:

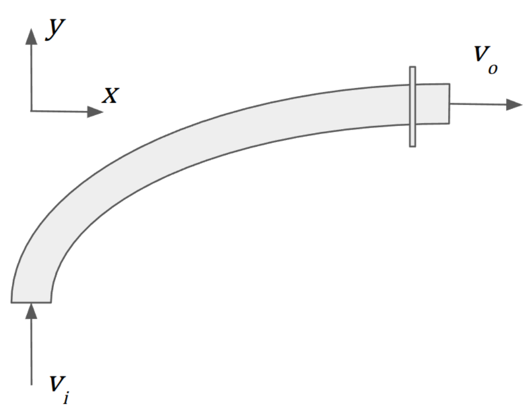

Figure 1: Control volume.

$$\sum \vec{F} = \frac{\partial}{\partial t} \int_{CV} \vec{v} \rho d\forall + \int_{CS} \vec{v} \rho (\vec{V} \cdot d\vec{A}). \tag{3}$$

In the x direction, we can simplify it to:

$$F_x = \dot{m}_o v_o. \tag{4}$$

This is the force we exert on the CV. The counter-reacting force we experience is $F_x' = -Fx = -\dot{m}_o v_o$, which pushes us to the left.

Question 3

The correct answer is **c**. The flow is shown in Fig 2.

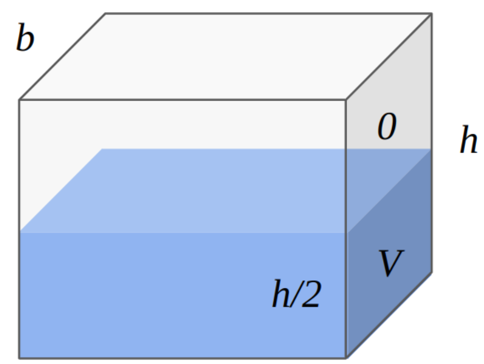

Figure 2: A half-filled duct pipe.

The definition of the kinetic energy correction factor α is the ratio between the actual energy flux across the surface and the energy flux calculated from the mean velocity on the surface:

$$\alpha = \frac{1}{A} \int_A \left(\frac{V}{\overline{\overline{V}}} \right)^3 dA \tag{5}$$

Since only the lower half of the duct is filled with fluid, we find that $\overline{V} = 0.5V$. Furthermore, the integration over the upper half of the cross-section is zero. Therefore, we have:

$$\alpha = \frac{A/2}{A} \left(\frac{V}{0.5V} \right)^3 = 4.0. \tag{6}$$

Question 4

- (a) *The energy line will be horizontal or slipping upward in the direction of flow.*
  This statement is **false**. In real flows, the energy carried by the flow will gradually dissipate into heat as it moves further due to surface resistance. So the energy grade line (EGL) always trends downwards, unless a pump injects the energy back into the flow externally.

- (b) *The energy line can never be horizontal or slopping upward in the direction of the flow.*
  This statement is **true**. The only case when an EGL moves upwards is with the help of a pump. However, we have specified that no additional source of energy is introduced in the system.

- (c) *The Piezometric line can never be horizontal or slopping* ***upward*** *in the direction of the flow.*
  This statement is **false**. The Piezometric line, a.k.a the hydraulic grade line (HGL), **can rise upward** in real flows. For example, at the gradually expanding entry from a pipe to a reservoir, the gradual expansion allows kinetic energy to be converted to pressure head with much smaller h_L. Hence the HGL tilts upwards and converges to EGL at the free surface.

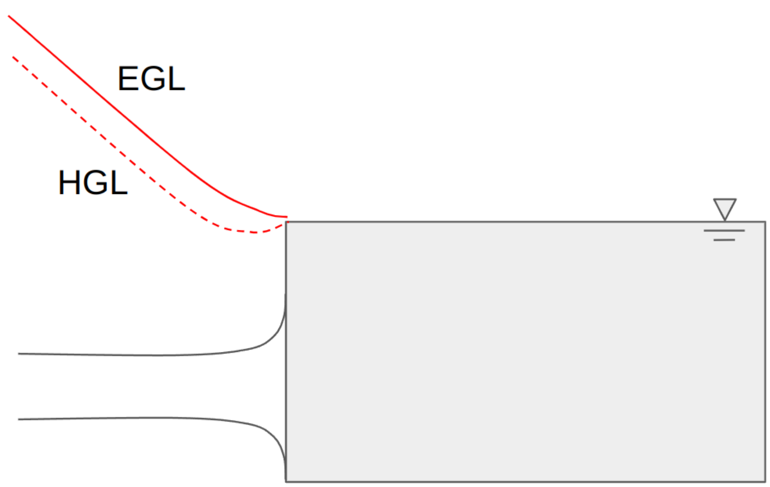

Figure 3: A gradually expanding entry.

- (d) *The center line of the pipe can never be above the Piezometric line.* This statement is **false**. A counter-example is a siphon system, where the centreline of the flow can be above the Piezometric line. In this case, the pressure levels in some parts of the pipe become subatmospheric, and cavitation could happen there.

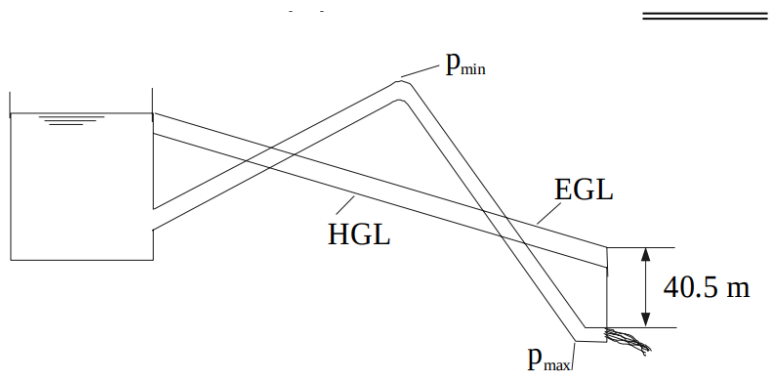

Figure 4: A siphon system.

Question 5

The cart will not move. If we build a control volume around the cart, we can see that there is no momentum flux coming in or coming out of the CV. Therefore, no force will be exerted on the cart.

Problem 1

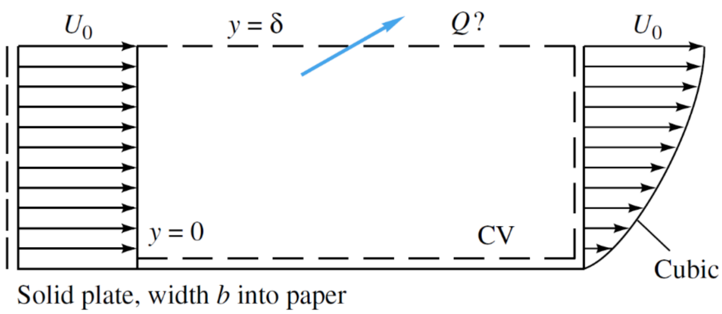

Figure 5: Control volume.

To solve this problem, we consider the conservation of mass in the control volume (Fig 5):

$$\frac{\partial}{\partial t}\int_{CV} \rho d\mathcal{V} + \int_{CS} \rho(\vec{V} \cdot d\vec{A}) = 0. \quad (7)$$

Since the flow is steady and incompressible, we can simplify it to:

$$\int_{CS} \vec{V} \cdot d\vec{A} = 0. \quad (8)$$

In other words, we only need to consider the balance of volume flow rates at different surfaces of the CV. The inlet is on the left surface, and the outlets are on the right and top surfaces:

$$Q_{left} = Q_{right} + Q_{top}. \quad (9)$$

At the left inlet with a uniform profile, we have

$$Q_{left} = U_0 b\delta. \quad (10)$$

At the right inlet, we need to integrate the velocity profile over the height:

$$Q_{right} = \int_0^{\delta} \frac{U_0 b}{2}\left(\frac{3y}{\delta} - \frac{y^3}{\delta^3}\right)dy = \frac{5}{8}U_0 b\delta \quad (11)$$

Therefore, we have

$$Q_{top} = U_0 b\delta - \frac{5}{8}U_0 b\delta = \frac{3}{8}U_0 b\delta. \quad (12)$$

Problem 2

To solve this problem, we build the control volume shown in Fig 6.

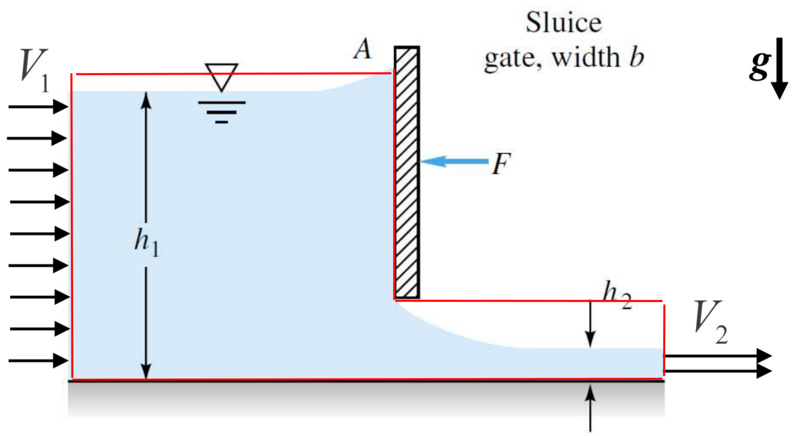

Figure 6: Control volume.

We first apply the conservation of mass to this CV:

$$\frac{\partial}{\partial t}\int_{CV}\rho d\forall + \int_{CS}\rho(\vec{V}\cdot d\vec{A}) = 0. \tag{13}$$

$$\dot{m}_1 = \dot{m}_2. \tag{14}$$

$$V_1 h_1 = V_2 h_2. \tag{15}$$

$$V_2 = V_1\frac{h_1}{h_2}. \tag{16}$$

Secondly, we apply the conservation of momentum:

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_{CV}\vec{v}\rho d\forall + \int_{CS}\vec{v}\rho(\vec{V}\cdot d\vec{A}). \tag{17}$$

Since we are solving a horizontal force F, we only consider the x direction of the equation and apply the uniform flow simplification:

$$\sum \vec{F_x} = \dot{m}_2 v_2 - \dot{m}_1 v_1. \tag{18}$$

We then identify the x-direction external forces acting on the CV (Fig 7). Notice that here we need to consider the hydro-static force acting on the inlet and outlet surface of CV due to the consideration of gravity. The formula for

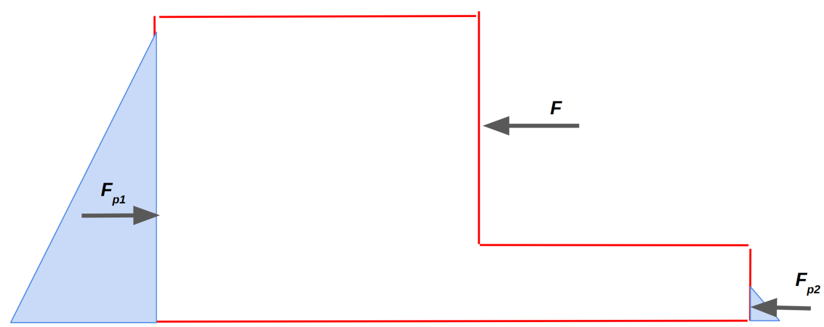

Figure 7: External forces.

the hydrostatic force acting on a vertical surface with a submerged depth of h and a width of b is:

$$F_p = \frac{1}{2}\rho g h^2 b \tag{19}$$

Therefore, we have

$$\sum \vec{F_x} = \frac{1}{2}\rho g h_1^2 b - \frac{1}{2}\rho g h_2^2 b - F. \tag{20}$$

Now we consider the RHS of the momentum equations:

$$\dot{m}_2 v_2 - \dot{m}_1 v_1 = \dot{m}_1(v_2 - v_1) = \rho v_1^2 h_1 b(\frac{h_1}{h_2} - 1). \tag{21}$$

Therefore, the external force needed to hold the gate is

$$F = \frac{1}{2}\rho g h_1^2 b - \frac{1}{2}\rho g h_2^2 b - \rho v_1^2 h_1 b(\frac{h_1}{h_2} - 1) \approx 333.4kN. \tag{22}$$

Problem 3

To solve this problem, we define a moving CV fixed on the cart:

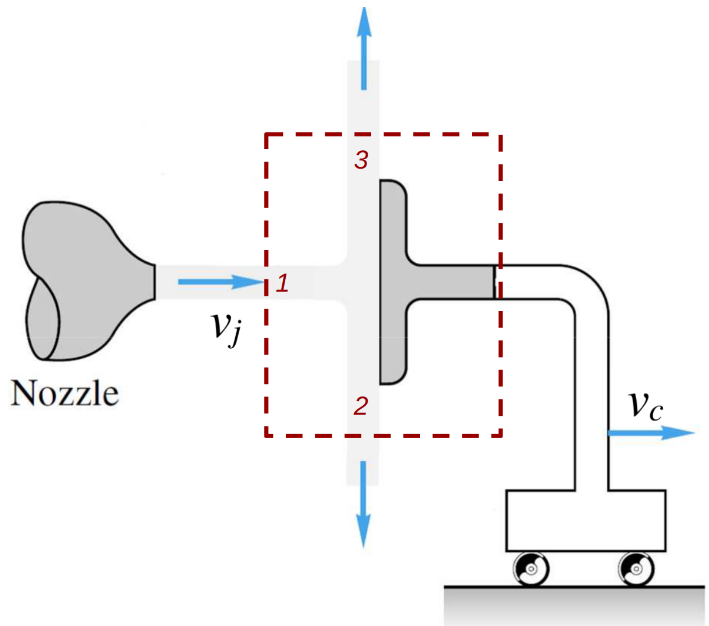

Figure 8: Control volume.

In the first step, we apply the conservation of mass:

$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho(\vec{V}\cdot d\vec{A}) = 0. \tag{23}$$

$$\dot{m}_1 = \dot{m}_2 + \dot{m}_3. \tag{24}$$

Notice that although our reference system is moving with the cart, **it is still an inertial reference system** as long as the speed of the cart is constant. Therefore, we have:

$$v_1 A_1 = v_2 A_2 + v_3 A_3. \tag{25}$$

$$A_2 = A_3 = 0.5 A_1. \tag{26}$$

Since the two small jets are equal, we have

$$v_2 = v_3. \tag{27}$$

We find that

$$v_1 = v_2 = v_3 = v_j - v_c. \tag{28}$$

Notice that v_1 is the jet velocity **we observe on the moving cart**.

Now, we can apply the momentum equations to compute the force needed to act on the cart to keep its speed constant.

In the x direction, we have

$$F_x = -\dot{m}_1 v_1 = -\rho(v_j - v_c)^2 A_j = -7.5N. \quad (29)$$

In the y direction, we have

$$F_y = \dot{m}_2 v_2 + \dot{m}_3 v_3 = \rho(v_j - v_c)^2 A_j/2 - \rho(v_j - v_c)^2 A_j/2 = 0N. \quad (30)$$

Problem 4

- (a) During the daytime, the water flows from reservoir 1 to 2, so the energy equation from point 1 to point 2 is:

$$z_1 + \alpha \frac{v_1^2}{2g} + h_p = z_2 + \alpha \frac{v_2^2}{2g} + h_t + h_L. \quad (31)$$

  Both reservoirs are still and open to the atmosphere, thus $v_1 = v_2 = 0$ m/s, $p_1 = p_2 = 0$ Pa, $\alpha \approx 1$. The pump is not operating during the day, so $h_p = 0$ m. We simplify the energy equation into:

$$z_1 = z_2 + h_t + h_L. \quad (32)$$

  We find that the turbine head $h_t = 37$ m, and the energy extracted from the flow by the turbine is $P_t = \dot{m} g h_t \approx 363$ kW.

- (b) Since the efficiency factor of the turbine is 0.8, the power generated by the turbine is $P_{gen} = 0.8 P_t \approx 290$ kW.

- (c) The HGL and EGL during the daytime are shown in Fig 9.

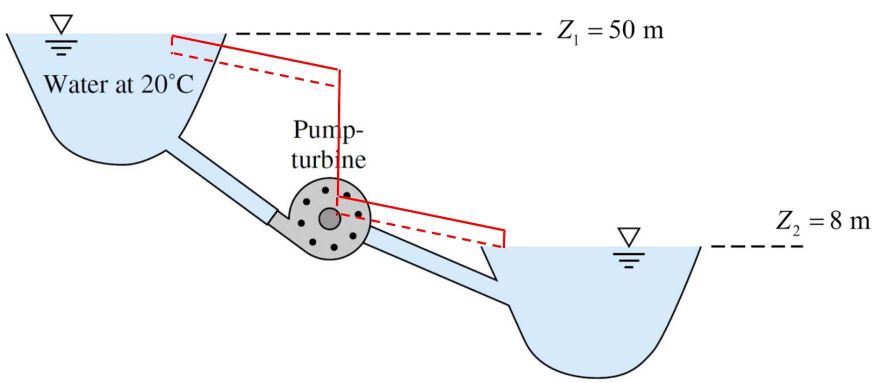

Figure 9: Daytime EGL (solid line) and HGL (dashed line).

- (d) During the night, the pump operates, and the turbine stops. The simplified energy equation from point 2 to point 1 is

$$z_2 + h_p = z_1 + h_L. \quad (33)$$

  We find that $h_p = 47$ m, and the power exerted by the pump to the water is $P_{pump} = \dot{m} g h_p \approx 461$ kW.

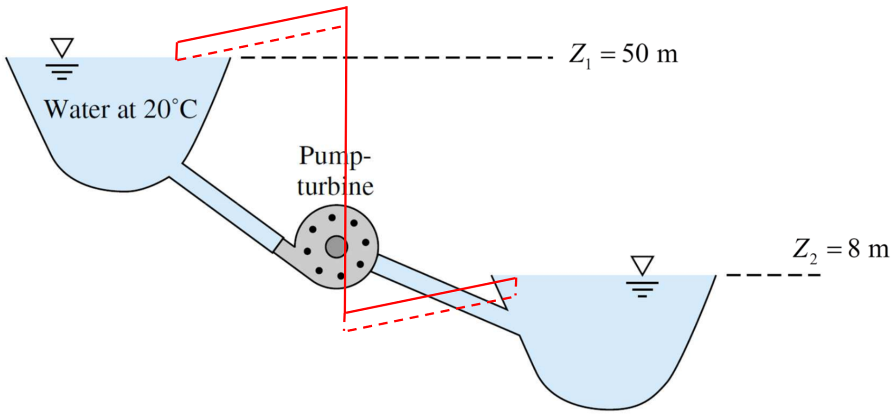

Figure 10: Night time EGL (solid line) and HGL (dashed line).

- (e) The efficiency factor of the pump is 0.8, so the electric power consumed by the pump is $P_e = P_{pump}/0.8 \approx 576$ kW.
- (f) The HGL and EGL during the night time are shown in Fig 10.